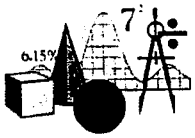
THE CATHOLIC UNIVERSITY OF AMERICA



First Annual CUA Newton Mathematics Competition 2005

Isaac Newton (1642-1727)

Team Competition

Problem 1. Solve for x the equation

$$\arcsin(3x) + \arcsin(x) = \frac{\pi}{2}.$$

Problem 2. Show that the sum of the fractions

$$\frac{a-b}{1+ab}, \frac{b-c}{1+bc}, \frac{c-a}{1+ca}$$

is equal to their product.

Problem 3. Given a right triangle ABC with right angle at C. The inscribed circle is tangent to AB at T. Prove that the area of the triangle ABC is equal to the product of the lengths of AT and TB.

Problem 4. How many digits does the integer $8^{102} \cdot 5^{302}$ have?

Problem 5. Every square of a 25×25-square table contains either the number 1 or -1. Let a_1 denote the product of all numbers in the first row, a_2 denote the product of all numbers in the second row, etc., a_{25} denote the product of all numbers in the 25-th row. Similarly, let b_1 denote the product of all numbers in the first column, b_2 denote the product of all numbers in the second column, etc., b_{25} denote the product of all numbers in the 25-th column. Prove that the sum $a_1 + a_2 + \cdots + a_{25} + b_1 + b_2 + \cdots + b_{25}$ cannot be equal to zero.

Problem 6. The *diameter* of a set in the plane is defined to be the maximal distance between two (2) points of the set. (For example, the diameter of a circle of radius 1 is 2, the diameter of a rectangle with sides 3 and 4, is 5.) What is the least positive number d such that a disc of radius 1 can be cut into 7 pieces of diameter d? (Let us recall that a disc of radius 1 consists of all points inside a circle of radius 1.)

Problem 7. Assume $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$, does not have real solutions ($p(x)$ is a polynomial with real coefficients). Show that there exist another polynomial $q(x)$ such that all coefficients of $p(x)q(x)$ are non-negative.